Method of Fractional Variation of Parameter and Its Applications

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Abstract: **Based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus, this paper studies the method of fractional variation of parameter. The product rule for fractional derivatives and a new multiplication of fractional analytic functions play important roles in this article. In addition, two examples are provided to illustrate how to use the method of fractional variation of parameter to find the particular solution of fractional differential equations. In fact, our results are generalizations of these results in ordinary differential equations.**

Keywords: **Jumarie's modified R-L fractional calculus, Method of fractional variation of parameter, Product rule, New multiplication, Fractional analytic functions, Particular solution, Fractional differential equations.**

I. INTRODUCTION

Fractional calculus comes from the generalization of differential and integral operators, which are applied to non- integer orders. In the past decades, fractional calculus has been widely used in quantum mechanics, electronic engineering, viscoelasticity, control theory, dynamics, economics and other fields [1-6]. However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [7-11]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on the Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, the method of fractional variation of parameter is studied. The product rule for fractional derivatives plays an important role in this article. Moreover, we give some examples to illustrate how to use the method of fractional variation of parameter to find the particular solution of fractional differential equations. In fact, these results we obtained are generalizations of those results in ordinary differential equations.

II. PRELIMINARIES

First, we introduce the fractional calculus used in this paper.

Definition 2.1 ([12])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ )$ is the gamma function. Moreover, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, and it is called the n-th order α-fractional derivative of $f(x)$, where n is any positive integer.

In the following, some properties of Jumarie type of fractional derivative are introduced.

Proposition 2.2 ([13])**:** *If* α,β,x_0,C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([14])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([15])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{8}$$

In the following, the arbitrary power of fractional analytic function is defined.

Definition 2.5 ([15])**:** Suppose that $0 < \alpha \leq 1$ and r is any real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes r} = E_\alpha\left(rLn_\alpha\left(f_\alpha(x^\alpha)\right)\right). \tag{9}$$

Definition 2.6 ([16])**:** Assume that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\big(g_\alpha(x^\alpha)\big)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\big(f_\alpha(x^\alpha)\big)^{\otimes k}. \tag{13}$$

Definition 2.7 ([16])**:** Suppose that $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

The α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$**.** In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

Moreover,

$$sec_\alpha(x^\alpha) = \big(cos_\alpha(x^\alpha)\big)^{\otimes -1} \tag{17}$$

is called the α-fractional secant function.

$$csc_\alpha(x^\alpha) = \big(sin_\alpha(x^\alpha)\big)^{\otimes -1} \tag{18}$$

is the α-fractional cosecant function.

$$tan_\alpha(x^\alpha) = sin_\alpha(x^\alpha) \otimes sec_\alpha(x^\alpha) \tag{19}$$

is the α-fractional tangent function. And

$$cot_\alpha(x^\alpha) = cos_\alpha(x^\alpha) \otimes csc_\alpha(x^\alpha) \tag{20}$$

is the α-fractional cotangent function.

Theorem 2.8 (product rule for fractional derivatives) ([17])**:** *Let* $0 < \alpha \le 1$, *and* $f_\alpha(x^\alpha)$ *and* $g_\alpha(x^\alpha)$ *be* α*-fractional analytic at* $x = x_0$, *then*

$$\left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{21}$$

III. MAJOR RESULT

In this section, we introduce the method of fractional variation of parameter.

Theorem 3.1: *Let* $0 < \alpha \le 1$, *and* $a_\alpha(x^\alpha), b_\alpha(x^\alpha), c_\alpha(x^\alpha), r_\alpha(x^\alpha)$ *be* α*-fractional analytic at* $x = 0$, $a_\alpha(x^\alpha) \ne 0$ *for all* x. *If the second order* α*-fractional differential equation*

$$a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] + b_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] + c_\alpha(x^\alpha)\otimes y_\alpha(x^\alpha) = r_\alpha(x^\alpha)\,. \tag{22}$$

has the homogeneous solution

$$y_{h,\alpha}(x^\alpha) = C_1 y_{1,\alpha}(x^\alpha) + C_2 y_{2,\alpha}(x^\alpha). \tag{23}$$

Then the particular solution of this α*-fractional differential equation is*

$$\begin{aligned} y_{p,\alpha}(x^\alpha) = &-\left({}_0I_x^\alpha\right)\left[y_{2,\alpha}(x^\alpha)\otimes r_\alpha(x^\alpha)\otimes\left\{a_\alpha(x^\alpha)\otimes W_\alpha\left(y_{1,\alpha}(x^\alpha), y_{2,\alpha}(x^\alpha)\right)\right\}^{\otimes -1}\right]\otimes y_{1,\alpha}(x^\alpha) \\ &+\left({}_0I_x^\alpha\right)\left[y_{1,\alpha}(x^\alpha)\otimes r_\alpha(x^\alpha)\otimes\left\{a_\alpha(x^\alpha)\otimes W_\alpha\left(y_{1,\alpha}(x^\alpha), y_{2,\alpha}(x^\alpha)\right)\right\}^{\otimes -1}\right]\otimes y_{2,\alpha}(x^\alpha). \end{aligned} \tag{24}$$

Where C_1, C_2 are constants and $W_\alpha\left(y_{1,\alpha}(x^\alpha), y_{2,\alpha}(x^\alpha)\right)$ is the α-fractional Wronskian of $y_{1,\alpha}(x^\alpha)$ and $y_{2,\alpha}(x^\alpha)$ defined by

$$W_\alpha\left(y_{1,\alpha}(x^\alpha), y_{2,\alpha}(x^\alpha)\right)$$

$$= \begin{vmatrix} y_{1,\alpha}(x^\alpha) & y_{2,\alpha}(x^\alpha) \\ \left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] & \left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)] \end{vmatrix}_\otimes$$

$$= y_{1,\alpha}(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)] - y_{2,\alpha}(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)]. \tag{25}$$

Proof By the method of fractional variation of parameter, let the particular solution of this α-fractional differential equation be

$$y_{p,\alpha}(x^\alpha) = p_\alpha(x^\alpha)\otimes y_{1,\alpha}(x^\alpha) + q_\alpha(x^\alpha)\otimes y_{2,\alpha}(x^\alpha). \tag{26}$$

Where $p_\alpha(x^\alpha)$ and $q_\alpha(x^\alpha)$ are α-fractional analytic functions. Then by product rule for fractional derivatives,

$$\left({}_0D_x^\alpha\right)[y_{p,\alpha}(x^\alpha)]$$

$$= p_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + q_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)]$$

$$+\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha). \tag{27}$$

And

$$\left({}_0D_x^\alpha\right)^2[y_{p,\alpha}(x^\alpha)]$$

$$= p_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_{1,\alpha}(x^\alpha)] + q_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_{2,\alpha}(x^\alpha)]$$

$$+\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)]$$

$$+\left({}_0D_x^\alpha\right)\left[\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha)\right]. \tag{28}$$

Since

$$a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_{p,\alpha}(x^\alpha)] + b_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{p,\alpha}(x^\alpha)] + c_\alpha(x^\alpha)\otimes y_{p,\alpha}(x^\alpha) = r_\alpha(x^\alpha). \tag{29}$$

It follows that

$$p_\alpha(x^\alpha)\otimes\left\{a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_{1,\alpha}(x^\alpha)] + b_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + c_\alpha(x^\alpha)\otimes y_{1,\alpha}(x^\alpha)\right\}$$

$$+q_\alpha(x^\alpha)\otimes\left\{a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[y_{1,\alpha}(x^\alpha)] + b_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + c_\alpha(x^\alpha)\otimes y_{1,\alpha}(x^\alpha)\right\}$$

$$+ a_\alpha(x^\alpha)\otimes\{\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)]\}$$

$$+ a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)\left[\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha)\right]$$

$$+ b_\alpha(x^\alpha)\otimes\{\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha)\} = r_\alpha(x^\alpha). \tag{30}$$

And hence

$$a_\alpha(x^\alpha)\otimes\{\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)]\}$$

$$+ a_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)\left[\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha)\right]$$

$$+ b_\alpha(x^\alpha)\otimes\{\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha)\} = r_\alpha(x^\alpha). \tag{31}$$

We can choose suitable $p_\alpha(x^\alpha)$ and $q_\alpha(x^\alpha)$ such that

$$\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes y_{1,\alpha}(x^\alpha) + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes y_{2,\alpha}(x^\alpha) = 0, \tag{32}$$

$$\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{1,\alpha}(x^\alpha)] + \left({}_0D_x^\alpha\right)[q_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)[y_{2,\alpha}(x^\alpha)] = r_\alpha(x^\alpha)\otimes\left(a_\alpha(x^\alpha)\right)^{\otimes -1}. \tag{33}$$

Thus,

$$\left({}_{0}D_{x}^{\alpha}\right)[p_{\alpha}(x^{\alpha})] = -y_{2,\alpha}(x^{\alpha})\otimes r_{\alpha}(x^{\alpha})\otimes\left\{a_{\alpha}(x^{\alpha})\otimes W_{\alpha}\left(y_{1,\alpha}(x^{\alpha}), y_{2,\alpha}(x^{\alpha})\right)\right\}^{\otimes -1}, \tag{34}$$

$$\left({}_{0}D_{x}^{\alpha}\right)[q_{\alpha}(x^{\alpha})] = y_{1,\alpha}(x^{\alpha})\otimes r_{\alpha}(x^{\alpha})\otimes\left\{a_{\alpha}(x^{\alpha})\otimes W_{\alpha}\left(y_{1,\alpha}(x^{\alpha}), y_{2,\alpha}(x^{\alpha})\right)\right\}^{\otimes -1}. \tag{35}$$

Therefore, we can choose

$$p_{\alpha}(x^{\alpha}) = -\left({}_{0}I_{x}^{\alpha}\right)\left[y_{2,\alpha}(x^{\alpha})\otimes r_{\alpha}(x^{\alpha})\otimes\left\{a_{\alpha}(x^{\alpha})\otimes W_{\alpha}\left(y_{1,\alpha}(x^{\alpha}), y_{2,\alpha}(x^{\alpha})\right)\right\}^{\otimes -1}\right], \tag{36}$$

$$q_{\alpha}(x^{\alpha}) = \left({}_{0}I_{x}^{\alpha}\right)\left[y_{1,\alpha}(x^{\alpha})\otimes r_{\alpha}(x^{\alpha})\otimes\left\{a_{\alpha}(x^{\alpha})\otimes W_{\alpha}\left(y_{1,\alpha}(x^{\alpha}), y_{2,\alpha}(x^{\alpha})\right)\right\}^{\otimes -1}\right]. \tag{37}$$

Finally, the particular solution of this α-fractional differential equation is obtained. Q.e.d.

IV. EXAMPLES

In the following, two examples are provided to illustrate how to use the method of fractional variation of parameter to obtain the particular solution of fractional differential equations.

Example 3.1: If $0 < \alpha \le 1$. Find the general solution of second order α-fractional differential equation

$$\left({}_{0}D_{x}^{\alpha}\right)^{2}[y_{\alpha}(x^{\alpha})] + y_{\alpha}(x^{\alpha}) = tan_{\alpha}(x^{\alpha}). \tag{38}$$

Solution Since the characteristic equation is $\lambda^2 + 1 = 0$. It follows that $\lambda = \pm i$. And hence, the homogeneous solution is

$$y_{h,\alpha}(x^{\alpha}) = C_1 cos_{\alpha}(x^{\alpha}) + C_2 sin_{\alpha}(x^{\alpha}). \tag{39}$$

Where C_1, C_2 are constants. Since the α-fractional Wronskian

$$\begin{aligned}
& W_{\alpha}\left(cos_{\alpha}(x^{\alpha}), sin_{\alpha}(x^{\alpha})\right) \\
&= \begin{vmatrix} cos_{\alpha}(x^{\alpha}) & sin_{\alpha}(x^{\alpha}) \\ \left({}_{0}D_{x}^{\alpha}\right)[cos_{\alpha}(x^{\alpha})] & \left({}_{0}D_{x}^{\alpha}\right)[sin_{\alpha}(x^{\alpha})] \end{vmatrix}_{\otimes} \\
&= \begin{vmatrix} cos_{\alpha}(x^{\alpha}) & sin_{\alpha}(x^{\alpha}) \\ -sin_{\alpha}(x^{\alpha}) & cos_{\alpha}(x^{\alpha}) \end{vmatrix}_{\otimes} \\
&= 1.
\end{aligned} \tag{40}$$

By Theorem 3.1, we obtain the particular solution

$$\begin{aligned}
& y_{p,\alpha}(x^{\alpha}) \\
&= -cos_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[sin_{\alpha}(x^{\alpha})\otimes tan_{\alpha}(x^{\alpha})] + sin_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[cos_{\alpha}(x^{\alpha})\otimes tan_{\alpha}(x^{\alpha})] \\
&= -cos_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[[sin_{\alpha}(x^{\alpha})]^{\otimes 2}\otimes[cos_{\alpha}(x^{\alpha})]^{\otimes -1}] + sin_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[sin_{\alpha}(x^{\alpha})] \\
&= -cos_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[\{1 - [cos_{\alpha}(x^{\alpha})]^{\otimes 2}\}\otimes[cos_{\alpha}(x^{\alpha})]^{\otimes -1}] + sin_{\alpha}(x^{\alpha})\otimes[-cos_{\alpha}(x^{\alpha}) + 1] \\
&= -cos_{\alpha}(x^{\alpha})\otimes\left({}_{0}I_{x}^{\alpha}\right)[sec_{\alpha}(x^{\alpha}) - cos_{\alpha}(x^{\alpha})] - sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(x^{\alpha}) + sin_{\alpha}(x^{\alpha}) \\
&= -cos_{\alpha}(x^{\alpha})\otimes\{Ln_{\alpha}(|sec_{\alpha}(x^{\alpha}) + tan_{\alpha}(x^{\alpha})|) - sin_{\alpha}(x^{\alpha})\} - sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(x^{\alpha}) + sin_{\alpha}(x^{\alpha}) \\
&= -cos_{\alpha}(x^{\alpha})\otimes Ln_{\alpha}(|sec_{\alpha}(x^{\alpha}) + tan_{\alpha}(x^{\alpha})|) + sin_{\alpha}(x^{\alpha}).
\end{aligned} \tag{41}$$

Finally, the general solution of this α-fractional differential equation is

$$\begin{aligned}
& y_{\alpha}(x^{\alpha}) \\
&= y_{h,\alpha}(x^{\alpha}) + y_{p,\alpha}(x^{\alpha}) \\
&= A\cdot cos_{\alpha}(x^{\alpha}) + B\cdot sin_{\alpha}(x^{\alpha}) - cos_{\alpha}(x^{\alpha})\otimes Ln_{\alpha}(|sec_{\alpha}(x^{\alpha}) + tan_{\alpha}(x^{\alpha})|).
\end{aligned} \tag{42}$$

Where A, B are constants.

Example 3.2: Let $0 < \alpha \leq 1$. Solve the second order α-fractional differential equation

$$\left({}_{0}D_{x}^{\alpha}\right)^{2}[y_{\alpha}(x^{\alpha})] - 6 \cdot \left({}_{0}D_{x}^{\alpha}\right)[y_{\alpha}(x^{\alpha})] + 9 \cdot y_{\alpha}(x^{\alpha}) = E_{\alpha}(2x^{\alpha})\,. \tag{43}$$

Solution The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$. Thus, $\lambda_1 = \lambda_2 = 3$. So, the homogeneous solution is

$$y_{h,\alpha}(x^{\alpha}) = C_1 \cdot E_{\alpha}(3x^{\alpha}) + C_2 \cdot \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}). \tag{44}$$

Where C_1, C_2 are constants. Since the α-fractional Wronskian

$$\begin{aligned} & W_{\alpha}\left(E_{\alpha}(3x^{\alpha}), \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha})\right) \\ &= \begin{vmatrix} E_{\alpha}(3x^{\alpha}) & \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \\ \left({}_{0}D_{x}^{\alpha}\right)[E_{\alpha}(3x^{\alpha})] & \left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha})\right] \end{vmatrix}_{\otimes} \\ &= \begin{vmatrix} E_{\alpha}(3x^{\alpha}) & \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \\ 3 \cdot E_{\alpha}(3x^{\alpha}) & E_{\alpha}(3x^{\alpha}) + 3 \cdot \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \end{vmatrix}_{\otimes} \\ &= E_{\alpha}(6x^{\alpha}). \end{aligned} \tag{45}$$

It follows that the particular solution is

$$\begin{aligned} y_{p,\alpha}(x^{\alpha}) &= -\left({}_{0}I_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \otimes E_{\alpha}(2x^{\alpha}) \otimes E_{\alpha}(-6x^{\alpha})\right] \otimes E_{\alpha}(3x^{\alpha}) \\ &\quad + \left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}(3x^{\alpha}) \otimes E_{\alpha}(2x^{\alpha}) \otimes E_{\alpha}(-6x^{\alpha})] \otimes \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \\ &= -E_{\alpha}(3x^{\alpha}) \otimes \left({}_{0}I_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(-x^{\alpha})\right] + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \otimes \left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}(-x^{\alpha})] \\ &= -E_{\alpha}(3x^{\alpha}) \otimes \left[\left(-\frac{1}{\Gamma(\alpha+1)}x^{\alpha} - 1\right) \otimes E_{\alpha}(-x^{\alpha}) + 1\right] + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \otimes [-E_{\alpha}(-x^{\alpha}) + 1] \\ &= \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 1\right)E_{\alpha}(2x^{\alpha}) - E_{\alpha}(3x^{\alpha}) - \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(2x^{\alpha}) + \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) \\ &= \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} - 1\right] \otimes E_{\alpha}(3x^{\alpha}) + E_{\alpha}(2x^{\alpha})\,. \end{aligned} \tag{46}$$

And hence, the general solution of this α-fractional differential equation is

$$\begin{aligned} & y_{\alpha}(x^{\alpha}) \\ &= y_{h,\alpha}(x^{\alpha}) + y_{p,\alpha}(x^{\alpha}) \\ &= C_1 \cdot E_{\alpha}(3x^{\alpha}) + C_2 \cdot \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) + \left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} - 1\right] \otimes E_{\alpha}(3x^{\alpha}) + E_{\alpha}(2x^{\alpha}) \\ &= C \cdot E_{\alpha}(3x^{\alpha}) + D \cdot \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes E_{\alpha}(3x^{\alpha}) + E_{\alpha}(2x^{\alpha}). \end{aligned} \tag{47}$$

Where C, D are constants.

V. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, the method of fractional variation of parameter is studied. The product rule for fractional derivatives and a new multiplication of fractional analytic functions play important roles in this article. On the other hand, some examples are given to illustrate how to use the method of fractional variation of parameter to find the particular solution of fractional differential equations. In fact, our results are generalizations of these results in ordinary differential equations. In the future, we will continue to use this method to expand our research field to applied mathematics and fractional differential equations.

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